Learning Single-digit Combinations: Developing Important Mathematical Ideas

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Part I - The Case

This article proposes a different instructional approach and purpose for becoming fluent in recalling the single-digit combinations[1] in all four operations. Becoming fast and accurate in recalling these combinations by rote has been a longtime sacred cow in American education once students enter second grade. Many times, young children have been excluded from working in more enriched mathematical settings based on slow performance on timed tests assessing the capacity to recall these combinations. What I hope to persuade you in this article is, by taking an algebraic approach towards the rich mathematical ideas that underlay single-digit combinations, students develop effective strategies in deriving, along with the properties of operations, which are the exact strategies and concepts needed when operating efficiently with multi-digit numbers. *The emphasis is on strategy development* rather than rote memorization to fluidly recall combinations.

Take the following mathematical expression:

$$7 + 9$$

This is a combination many children take a long time committing to memory. Consider how posing the following number string to students might help them develop a strategy to effectively solve the sum of 7 + 9 en route to fluent recall (Fosnot & Dolk, 2001).

7 + 9 = 7 + 10 – 1	*True or False? Why true or why false?*
7 + 9 = 10 + 9 – ☐	*What needs to go in the box to make the number sentence true?*
7 + 9 = 7 + 7 + ☐	
7 + 9 = 6 + 10	*True or false? Why true or why false? Which expression would you rather work with?*
7 + 9 = ☐	*Which strategy from above can you use to figure out 7 + 9?*

Historically, teachers in the United States have sent the single-digit combinations home with students with the directive to memorize them, followed by timed tests to assess the students' capacity to simply recall each one individually. Teachers frequently complain that students who don't know their combinations fall back to counting on their fingers to determine the result. I have been in and around elementary classrooms for over three decades. This is an on-going complaint that precedes my time as a teacher. I fully agree that the more fluent students are in recalling the single-digit combinations the easier time they have when operating with multi-digit numbers. The professional question is, how do we get children to attain this level of recall?

A Learning Theory Perspective

A little visit with learning theory and what research has revealed is worth a moment's time. Researchers have demonstrated that children progress through developmental stages of understanding and strategy use in number operations (Carpenter, et al., 2015). Children progress through the stages of direct modeling (a concrete level where two blocks and five blocks are laid out then all the blocks are counted starting from one) and counting on (initially starting from 2 then counting on five more numbers; later on, being able to count on from the larger starting with five then counting two more numbers). The question is, what is the next developmental stage? To understand next progressions, let's revisit the notion of *'the zone of proximal development'* originally articulated by Vygotsky.

[1] I am using the term *single-digit combinations* rather than the term "basic facts" for two reasons. Single-digit combinations is more mathematically accurate and aligns with term *multi-digit strategies* that children encounter when needing to operate on numbers. The second reason is that the cultural term of *basic facts* is misleading and emotionally loaded. Others in the math education community are also using this terminology.

Think of what many elementary teachers have come to view regarding children's developmental progressions in literacy. The books children read independently are at their *comfort level*. That is different from the texts used at students' *instructional level* that is set at the student's next developmental stage. Read aloud literature is often a level ahead of the instructional level with the intent to develop children's *oral comprehension*. This 'zone of proximal development' that we use in literacy holds true in strategy development in mathematics. If I don't know 7 + 9 at a *recall level yet*, I will revert to my *comfort level* to solve the problem. If counting on by ones is the only other strategy that I know, then it is that level to which I will revert if I can't recall the combination. So, what is in between counting on and recall, or in the case of multiplication, skip counting and recall, if any?

Carpenter & Moser (1984) in their work with children articulated what some children start to do on their own; using *derived strategies.* Derived strategies are based on the mathematical principles of 'decomposition of number' and the use of 'relational thinking' by taking a known single-digit combination to figure out an unknown combination. The combinations posed earlier are examples of derived strategies. View the following video of a group of first graders working with a number string that has as its initial focus using the doubles plus and minus one strategy.

Pause and View this Video

[**First Grade Number String**]

There are several things to notice in this episode. Jamie clearly knows *intuitively* that 6 + 7 is thirteen and *intuitively* did so knowing that 6 + 6 + 1 is the same as 6 + 7. However, when asked *explicitly* to state where the other six was coming from, his ability to place into language his intuitive thoughts reached a limit. It was through the voice of Sophie making the decomposition of seven into 6 + 1 explicit that Jamie was able to continue explaining his thinking. The teacher's role of capturing the student's thinking in a combination of informal and formal notation allows the students to consider the trace of their thinking, giving a visual form to wrap their voice around.

Further on in the episode, even though the instructional intent was to foster a conversation around the doubles plus or minus one strategy, Grace articulates a *Make a Ten* strategy. She decomposes one of the sixes into three plus three in order to combine 7 + 3 to create a ten, then follows by adding the other 3 to 10 to find the answer 13. What is intentional about the number string presented to the students is the goal of explicitly exploring and nurturing derived strategies en route to being able to recall these combinations. Therefore, if I don't automatically know the sum of 7 + 9 *yet,* and *if I have been encouraged to make derived strategies my comfort zone*, instead of falling back on counting strategies, deriving becomes the fallback strategy. While it is developmentally normal for children to use counting strategies, they need to be *intentionally and purposefully* mentored and supported over time to develop deriving strategies. The mathematical processes used in deriving are far more advanced than those used in counting on.

Built on Foundational Fluency

Fluency in developing addition and subtraction derived strategies are built around a core set of foundational combinations (Bay-Williams & Kling 2019, Ma, 1999). These foundational combinations are the doubles (3+3; 6+6…), combinations of 10 (7+3, 6+4…), and the compensation strategies that are built on the core algebraic concept of zero plus a number in conjunction with the place value concept of 10+ a single number, e.g. 10 + 2 = 12. Additionally, a number minus itself in conjunction with the place value concept of a double-digit number minus its one's digit, e.g. 12 - 2 = 10 is key decomposition of number knowledge. See the attached Bay-Williams & Kling handout. Fluency with these core combinations allows students to engage in what Chinese educators express as learning "addition with composing and subtraction with decomposing within 20" (Ma, 1999, p. 16). Students use knowledge of these core combinations to then use derived strategies such as the *Doubles plus or minus, Make a Ten, and Compensation strategy of 10* to determine those single-digit combinations with sums above ten. Example:

I use…

- ☐ Doubles ±
  - ☐ 4 + 4 to figure out 5 + 4.
  - ☐ 7 + 7 to figure out 6 + 7
- ☐ Make a 10, Get back to a Ten

   - ☐ 7 + 3 to figure out 7 + 5
   - ☐ 12 - 2 to figure out 12 - 5
- ☐ Compensation
   - ☐ 10 + 6 to figure out 9 + 6
   - ☐ 10 - 4 to figure out 11 - 4

The strategies described below such as using number strings often start with what has previously become fluency with these foundational combinations. Fluency and the mathematical reasoning that arises with using such strategies in addition and subtraction carry over to students' capacity to develop fluency in multiplication and division. A student's core foundational fluency with squares and doubling, e.g., 3 x 3, helps derive 3 x 6. The distributive property of multiplication over addition which is built on the concept of configuring and decomposition of number allows a student to solve 7 x 6 as 5x6 + 2x6. Compensation allows one to use 10 x 6 to figure out 9 x 6.

What are the benefits?

So, why not continue to just have students go home and memorize these combinations? Isn't spending time on derived fact strategies just a waste of class time? As to the former, America has been asking students to memorize by rote recall for generations and it only works for the few. Too many students, who otherwise have decent math skills, do not succeed with this approach and end up with extreme anxiety issues around math in their daily lives. As to the latter response, becoming skilled in using derived strategies with single-digit combinations are the very strategies that students need to use to be fluent with when operating on multi-digit numbers. The strategies draw out the algebraic properties of operations as well. Therefore, deriving strategies develop transferable math skills useful for more sophisticated mathematics encountered later in their learning.

Consider this number string designed around the core strategy of *Make a Ten.*[2]

7 + 3	
7 + 5 = 7 + 3 + ☐	*How can you use what you know about 7 + 3 to help you figure out 7 + 5?*
17 + 5	*How can that same strategy used for 7 + 5 be used to solve 17 + 5?*
47 + 5	*[Getting to the next ten]*
7 + 5	*What is the sum of 7 + 5?*

Beyond developing a specific strategy to learn 7 + 5, students are developing fluency around decomposing numbers into relevant subparts, and using the associative property of addition. [47 + 5 = 47 + 3 + 2 = 50 + 2] They are also seeing how the relationships of strategies used for single-digit combinations can be used with multi-digit combinations. The ideas and skills are both connected and generative.

Fluency Standard

According to the research by Kamii (2000), a student who gives the total to a single-digit combination in two seconds or less is doing so at the recall level. If more than two seconds, the student is using some level of calculation. The question to be investigated with the student is what strategy is being used to calculate. Carpenter, et al. (2015) in their work have found that students who are fluent in recalling single-digit combinations relied on derived strategies to get to the recall level. Those derived strategies are so well practiced that the combinations are answered within two seconds or less. However, the student's explanation of how that particular combination was recalled may be an explanation of the derived strategy used in being able to recall quickly.

Interim Summary One

Let's summarize the reasoning behind a different approach towards learning the single-digit combinations with sums between 1 and 20. This position is based upon three core ideas from research. First is that if an individual is not able to

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[2] While exploring and developing the *Doubles ± One* strategy is useful, the '*Make a Ten'* (addition) and '*Get back to a Ten'* (subtraction) strategies are the most productive when working with multi-digit numbers.

instantly recall a particular combination, that person will revert to a comfort level strategy to calculate the answer. Due to most generational practices around learning these single-digit combinations here in the United States, that comfort level strategy is counting on or back on one's fingers by ones (addition) or skip counting starting from zero (multiplication). Developmental learning research is the second core idea. There is a developmental strategy level between counting on/back and being able to recall combinations automatically. That stage of development is derived strategies. This means that significant instructional time needs to be spent nurturing derived strategies with students at this level so that these new strategies become the students' new comfort level. Thirdly, these derived strategies are the very strategies students need to develop to be fluent in decomposing, reconfiguring, and operating on multi-digit combinations. Derived strategies draw out the algebraic properties and relational thinking around equivalencies that underlay the four operations. An instructional focus on derived strategies, therefore, develops multiple concepts and skills that students need as they progress through future levels of mathematics.

Figure 1: Developmental Levels of Children's Solution Strategies (Carpenter, et al. 2015)

Direct modeling ⟵⟶ Counting Strategies ⟵⟶ Derived Strategies ⟵⟶ Automatic Recall

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Reflection Point 1

Take a moment with your colleagues to summarize what has been said at this point, what was viewed in the video, and the linked article about organizing around landmarks of ten.

- ☐ What are the main benefits of spending time during class instruction on developing derived strategies with students?
- ☐ How does the approach to developing fluid recall compare to your current school community and textbook practices?
- ☐ Is organizing around landmarks of ten a new idea to you? How do you see students benefiting from being adept at using this type of number sense?

Part II - Developing Relational Thinking

Using Number Strings to Develop Derived Strategies

Number strings are *one method* of having focused conversations with students around derived strategies and the properties of operations (Fosnot & Dolk, 2001). Multiplication is featured here as the operation around which these number strings can be formed. The core ideas expressed here are readily extended to the other three operations. The ideas to ruminate on are,

a) How to initiate a string to gain maximum participation,
b) How to foster the relational thinking from one element of the string to the next,
c) How to capture student thinking through various forms of representation, and
d) How to use questions and prompts that elicits and clarifies student thinking.

To begin, view the video by clicking on the link.

Pause and View this Video

[Fourth grade number string]

As I have come to develop these sequences for my students (who now include adult learners), I begin with a combination that will gain maximum participation among the students. In this instance, the string began with the square 7 x 7. The product of square combinations such as 7 x 7 are often known before some of the neighboring combinations. Beginning with five combinations (x 5), doubles (x 2), and tens (x 10) are key examples of where to start in order to gain maximum participation. As the string unfolds, the point is to begin to stretch relationally to harder neighboring combinations. Next in the sequence was 7 x 9 as this is a combination that students take a long time in gaining automaticity. The initial response by Milda (64) is an example. Think about the products 48, 49, 54, 56, 63, and 64. Which combination(s) goes with which product is difficult for students to sort through if a strict rote memorization process is relied upon to recall each individual combination. Some teachers, as a result of well-intentioned reasons, teach children mnemonic devices and chants to recall specific combinations. Erika is an example of a student having been taught the 'nine fingers trick' to recall 7 x 9. While she gets a correct answer, she is not engaged in mathematics. That memory device is limited in scope and is not foundational for any future mathematics.

The prompt specifically posed to students was *If you know what 7 x 7 is, what would 7 x 9 be?* This prompt is instructionally aimed at strategy development whether or not you already know the combination at an automatic level. Such prompts foster relational thinking. This results in a network of related strategies allowing automaticity in recalling a product. As the conversation in the video unfolded, more than one strategy emerged, not just those based on 7 x 7. Nevertheless, 7 x 7 serves as a catalyst for those students who need to move off of skip counting beginning from zero.

Raleigh is able to articulate that seven nines are the same as having seven sevens plus two more sevens. Captured on the board by the teacher, this equation traces Raleigh's thinking: 7 x 9 = 7 x 7 + 7 x 2. One could further clarify his thinking by inserting the unspoken decomposition of nine being the same as seven plus two as in 7 x 9 = 7 x (7 + 2) = 7 x 7 + 7 x 2. That is an in-the-moment decision that you as the teacher need to make. What Raleigh demonstrated confidently, whether he is explicitly aware of it yet or not, is *the distributive property of multiplication over addition*. The representation serves not only for Raleigh as a means to clarify his thinking but serves as a means for others in the classroom to reflect visually upon his verbal descriptions and make mathematical sense of his work. Adding to the representation is a decision based on your reading the level of understanding that is evident among those listeners.

Ed follows Raleigh with *a compensation strategy* relating 7 x 9 to 7 x 10. Ed articulates that 7 x 9 = 7 x 10 – 7 x 1 as he describes that he needed to subtract "one, seven times" to end up with seven nines. The representation placed on the board in this episode shows 10 x 7 – 7. Whether or not to use "– 7" or "– 1 x 7" depends upon the novelty of the strategy among the class members, in which case 1 x 7 adds clarity. Or, if that has become collectively understood, "– 7" suffices. The decision is based upon your knowledge of the needs within the group. The compensation strategy is an example of *the distributive property of multiplication over subtraction*: 7 x 9 = 7 x (10 – 1) = 7 x 10 – 7 x 1

Compare the foundational use of the distributive property of multiplication over addition, and compensation strategy used by these two students above with the limited capacity of teaching mnemonic devices, e.g., Erika. To use an analogy based on the *Three Little Pigs,* Raleigh's and Ed's thinking allows each of them to build a house of brick, whereas a past teacher's good intentions to teach Erika a mnemonic device leads to a house of straw.

The third element in this string was 7 x 16. This is not a single-digit combination. However, extending a string to just outside the range of single-digit recall fosters the understanding that the strategies used to derive single-digit combinations are the very strategies needed to work with multi-digit combinations. This nurtures the relational thinking that underpins the associations among a network of mathematical ideas.

Devan responds with the strategy of 7 x 10 + 7 x 6, a use of the distributive property of multiplication over addition by decomposing the sixteen into its place value components (16 = 10 + 6). Celeste uses the information from the two previous expressions that combine to make sixteen; 7 x 16 = 7 x (7 + 9) = 7 x 7 + 7 x 9. Emma uses the first element in the string, 7 x 7, twice, then understands that she still needs two more sevens to combine for a total of 16 groups (16 = 7 + 7 + 2). Celeste's and Emma's strategies both represent the distributive property of multiplication over addition, just with different decompositions than Devan's. Both, however, were built relationally from previous pieces of information. Sachin uses the compensation strategy that is built around 7 x 20 – 7 x 4. He starts with seven twenties via knowing seven tens then doubling it.

The richness of strategies and the evidence of algebraic and relational thinking is strong among those students that shared in this video clip. The number string could have taken other directions. The formation of any future string you design, the main elements are what is the mathematical goal, how to start by gaining maximum participation, and how to begin to stretch the next items in the string to foster connections with larger numbers as well as developing relational thinking. While your string may be designed to foster a conversation around a particular strategy, being open to any and all strategies is essential. What relationship is evident to one student is not necessarily the relationship seen by another. The diversity of thinking is to be celebrated. The ultimate goal is to scaffold students off counting by ones or skip counting from zero by becoming comfortable with organizing around key landmarks. These derived strategies build the very foundation to then begin operating with multi-digit numbers. In essence, *use what you know to figure out what you don't know.* That is a key element for progressing further in mathematics.

Interim Summary Two

Developing these derived strategies is based upon specific developmental research on how to assist students in gaining fluency with these combinations. Rather than a sole emphasis on rote memorization in high stakes timed environments, consciously spending time on nurturing derived strategies helps students develop the very strategies and algebraic relations necessary when operating with multi-digit numbers. Example: 12 – 5. If I know 12 – 2 = 10, and 10 – 3 = 7, then 12 – 5 = 7. I can use the strategy of *Get back to a Ten* to fluently and accurately solve the problem. This is more efficient than counting back one-by-one. This strategy holds if the numbers were 72 – 15 as I can quickly increment backwards going 72 – 10 ⟶ 62 – 2 ⟶ 60 – 3 ⟶ 57. Students who are well grounded in such strategies lay down such strong neurological pathways that they determine the result within two to three seconds. This is the time considered to be within the recall range.

The same is true for multiplication single-digit combinations. Instead of teaching students the nines finger trick as a quick (albeit limited) mnemonic device, compensation is an effective mathematical strategy that extends to complex multi-digit combinations. Example: 9 x 7 is frequently one of the combinations students have difficulty committing to memory. Instead of skip counting by sevens starting from zero, 10 x 7 is an easily known combination that is readily recalled. So, 9 x 7 = 10 x 7 – 1 x 7 as in *Nine groups of seven is the same as having ten groups of seven minus one less group of seven.* This can extend to a combination such as 29 x 12. 29 x 12 = 30 x 12 – 1 x 12. The strategy is transferable. The number sense is strong. And, if the strategy has become well ingrained and fluent, the solving of more complex combinations become fluent and accurate as well.

Spending intentional and focused time on nurturing these strategies across the grades yields long term benefits. The strategies are grounded in algebraic properties. The strategies are generative – *pedagogically sustainable* – as they allow the student to grow mathematically in the future. The strategies build towards future mathematical concepts. The trouble with mnemonic devices and strict rote recall is that they have limited use. They only pertain to that one particular recalled item. This is the general issue with a number of tricks and surface patterns teachers introduce to students to "help" and speed things along. They may help students get through the short-term test but the tricks and devices limit the students' capacity to engage in deeper mathematical concepts later on. (Karp, et al. 2014)

Reflection Point 2

Take a moment with your colleagues to summarize what has been said in Part 2, what was viewed in the video, and the linked article about organizing around landmarks of ten.

- ☐ Language in the Common Core Standards consistently uses the phrase, "Apply properties of operations as strategies," and, "using strategies based on place value and properties of operations."
    - ☐ Given the previous section(s), what are the *properties of operations.*
    - ☐ How does spending time intentionally and purposefully nurturing derived strategies help develop those properties?
- ☐ Again, how does this perspective on developing fluid recall of single-digit combinations meld or not with current practices within your current learning community?

Part III - Designing Instructional Tasks

Designing Number Strings

The following are examples of how some simple number strings are designed to help students become fluent with certain multiplication single-digit combinations. The focus here is what research studies, along with decades of teacher observations, have noted to be remarkably hard for students in gaining fluid recall. The image below (See figure 2) comes from the work of Kamii & Livingston (1994). Notice that the one white area that captures 3x6 through 4x9 is also captured in the commuted combinations of 6x3 through 9x4. The third area captures the longest to commit to memory: 6x6 through 9x9. If, as a teacher, you are aware that these combinations are more difficult with which to gain fluency, then designing a series of number strings, true/false, and open number sentences presented over an extended period of time helps students build a network of related combinations that increases the capacity for fluid recall. Again, these strategies are transferable to multi-digit combinations. You are fostering both fluency, foundational algebraic properties, and relational thinking.

Learning Intent:

- Anchor 3x6 and 6x3 (Three groups of six and six groups of three), particularly to interrupting a tendency to skip count to find the answer
- Use these two combinations to derive related and more difficult combinations
- Nurture use of the distributive property of multiplication over addition as well as the commutative property

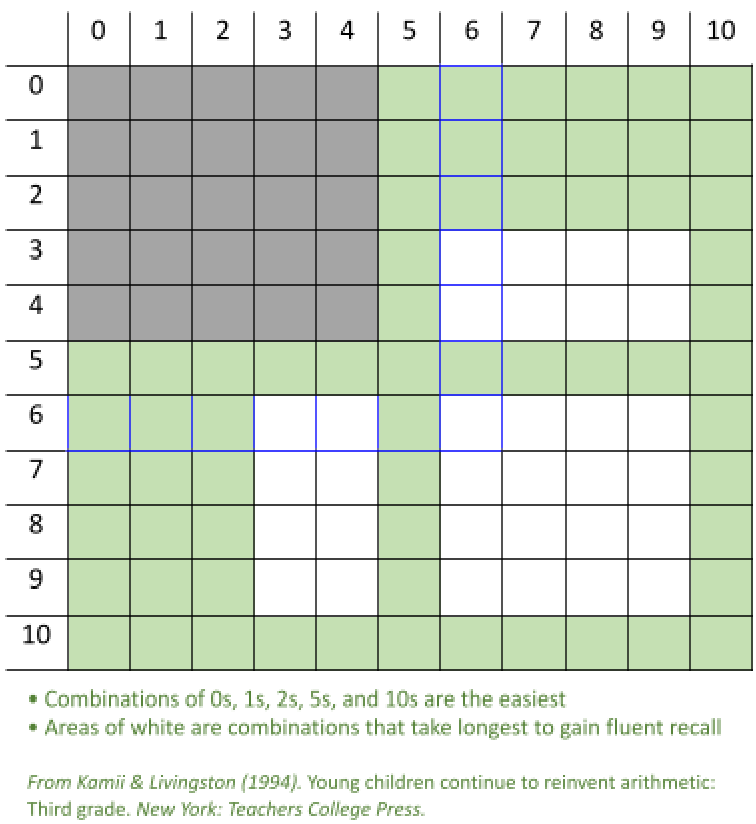

Figure 2: Sequence of recalled factor combinations

Posing and Asking Questions

Day 1: [*Show each expression one at a time*]

3 x 6[3]	*What are three groups of six?*
6 x 6	*Use what you know about three sixes to figure out six sixes.*
12 x 6	*Use what you know about six sixes or three sixes to figure out twelve sixes.*
3 x 6	*So again, what are three sixes?*
5 x 6	*Use what you know about three sixes to figure out five sixes.*
8 x 6	*Use these two previous combinations to figure out this new one.*

Day 2:

3 x 6	[Use similar language as previously]
4 x 6	
7 x 6	
3 x 6	
4 x 6	
8 x 6	

Day 3:

3 x 6	
6 x 3	Follow-up question: *Why does 3 x 6 have the same answer as 6 x 3? 3 x 6 = 6 x 3*
6 x 6	*Use either of the above combinations to figure out six sixes.*
6 x 3	
7 x 3	
8 x 3	

Day 4:

3 x 6
3 x 8
3 x 7

6 x 3
3 x 3
8 x 3
4 x 3

Day 5:

3 x 7
6 x 7
9 x 7

6 x 3
6 x 4
6 x 7

There is nothing particularly magical about the above sequences. The examples demonstrate how, in an opening warm-up at the beginning of a lesson, substantive conversation can happen around single-digit combinations and to use what you know about one combination to figure out another combination. Yes, you can skip count starting from zero each time – something I remind students frequently – but how can you use one combination to figure out other combinations to save yourself some time. The message that is continually conveyed is two-pronged: use relational

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[3] The convention being used in this article matches that used both in CGI research and in the Common Core Standards. In the expression *a x b,* '*a*' represents the *number of groups* and *'b'* represents the number of items within each group. *'a'* is the multiplier. When the '*times*' language convention is used, as in "*a times b," 'b'* becomes the multiplier as in "I want '*a*' things, '*b' times.*

thinking – Use what you know to figure out what you don't know – and work efficiently to save yourself time in the future. Recall, by itself, does not build these two key dispositions of relational thinking and making efficient judgements.

Capturing Student Thinking Mathematically in Number

More than asking students for the answer and verbally having them describe the strategy used to derive it, how you capture student's thinking mathematically will move the conversation towards exploring underlying algebraic principles. As a student begins explaining the thought process used to find a solution, I intentionally capture that information in very specific formats. The product or sum may or may not ever be written. Note how this was done in the two video clips previously viewed as part of this article. The following is one example of how Day 1 and Day 3 *might* unfold.

Day 1: [*Show each expression one at a time*]

3 x 6 — *What are three groups of six?*

- If the student builds off of 2 x 6 then I would *say* and *write*…

  3 x 6 = 2 x 6 + 1 x 6

  So, you are telling me that three groups of six is the same as having two groups of six plus another one group of six? Did I hear you correctly? [Yes] *And tell me again what the total of three sixes are.* [18]

6 x 6 — *Use what you know about three sixes to figure out six sixes.*

- *6 x 6 = 3 x 6 + 3 x 6*
  = 18 + 18
  = 36

  [T] *So, you used three groups of six and added another three groups of six, and you added the two eighteens together and that gave you a total of 36, is that correct?*

  [T] So, you broke the 6 into 3 + 3 to have a total of 6 groups of six.

  6 x 6
  */ *
  3 + 3

12 x 6 — *Use what you know about six sixes to figure out twelve sixes.*

- *12 x 6 = 6 x 6 + 6 x 6*
  = 36 + 36
  = 72
- *or – and you need to expect students to recognize alternative strategies…*
  12 x 6 = 10 x 6 + 2 x 6
  = 60 + 12

Notice the above notation uses a combination of very formal and informal notation. This is done to focus students' attention to the decomposition of the number. The second aspect of the notation is the first student called on for 6 x 6 said, *"I just added 18 plus 18 and got 36,"* I very specifically channel that statement to first write 3 x 6 + 3 x 6, saying, *"So, the eighteen came from using 3 groups of six and you then added another*. This editorial choice is an intentional decision to focus students' attention on the algebraic relationships of what eventually can become 6 x 6 = (3 + 3) x 6 = 3 x 6 + 3 x 6; otherwise known as the distributive property of multiplication over addition. *The notation also models for students how they, themselves, can begin to notate their thinking moving forward in their work. The use of parentheses, the proper use of the equal sign, and the order of operations are all subtly introduced as a means to express one's thinking rather than as rules and definitions to memorize.*

Day 3:

3 x 6 — *What are three groups of six?*
– Note if more hands go up quickly compared to the previous two days. This will be an indicator that this combination is gravitating to the recall level

6 x 3 — *What are six threes?*
Ask: *Why does 3 x 6 have the same answer as 6 x 3?* [writing out] *3 x 6 = 6 x 3; Yes, they have the same product, the same total, but why do they end up being the same? Six bags with three bagels inside each bag looks and feels different than 3 bags with six bagels inside each. Why do they turn out to be the same?*

Equal grouping contexts are *asymmetrical.* In the context of bagels described above, the commutativity aspect of the scenario is not as readily apparent to students as adults think. Spending time discussing the relationships of the equality, even to the point of working the context out with drawings or manipulatives so students can rearrange the objects is worthwhile doing. Arrays and area models are *symmetrical.* The commutative property is more easily seen. However, life in our culture is more dominated with packaging images – *asymmetrical contexts* – rather than things that come in neat rows and columns – *symmetrical contexts*. The time spent discussing the commutative property with single-digit combinations such as 3 x 6 and 6 x 3 is worthwhile so that when students encounter a problem such as 25 boxes of 6 (25 x 6) where the emphasis is on skip counting sixes, commuting the scenario to be 6 boxes of 25 (6 x 25) is a lot less work.

There is no one recipe for doing these number strings. [Experiment with designing some combination strings.] The goal is to build strategies, to assist students in understanding the underlying mathematics, to think relationally, and model how to capture their thinking in mathematical notation. This will help them transfer this knowledge to other contexts as they advance in their mathematical learning.

Reflection Point 3

Take a moment with your colleagues to summarize what has been said in Part 3 and the linked article about designing number strings.

- ☐ Look back at Figure 2, the multiplication single-digit combinations. What would be the equivalent, more difficult addition and subtraction combinations that students seem to take longer to gain fluency with?
    - ☐ What might a series of number strings look like to foster strategy development and fluid recall for those combinations?
- ☐ Role play with colleagues a set of number strings. Example: Verbally sharing the strategies used to solve the 17-times table. Practice capturing a colleague's strategy in number to give yourself an opportunity to consider the different ways that colleague's thinking can be represented in number.

Part IV - An Implementation Policy

Timed tests ⟵⟶ Assessment Goals

Assessing a student's recall of single-digit combinations inevitably comes around to how timed tests are used. This section seeks to have you rethink the role and purpose of timed assessments. As stated previously, I fully agree that the more fluid a child is in recalling single-digits combinations the more efficient one can be in solving mathematical tasks. *It is how we support students in developing this fluency that is the professional development point of conversation.* The emphasis on this approach to supporting students in learning the single-digit combinations is through the generative use of strategies that not only work with these smaller combinations but work with multi-digit strategies as well.

Timed tests are stressful for many students. Under stress, the brain tends to revert to using strategies that are at the individual's comfort level. Historically for most students, this means counting on/back by ones (addition & subtraction) or skip counting starting from zero (multiplication and division). Revisiting Vygotsky's theory articulating the concept of the *zone of proximal development,* (Figure 3) when scaffolded instructional support is removed, students revert to their comfort level. Traditionally, under a rote memorization/timed test regimen, that means counting on one's fingers if one can't recall that specific single, unique item. Where instruction supports the development of derived facts intentionally and actively over time, derived facts become the comfort zone rather than counting on. The underlying algebraic properties of operations along with the strategies needed to work with multi-digit numbers are simultaneously developed in tandem.

Figure 3: Comparison of student development under two instructional approaches towards recalling single-digit combinations.

Typical student reaction under a rote memorization/timed test regimen

• Direct modeling		
• Counting	Comfort Zone	
• Number recall	Instructional Zone	

Typical progression of student development over time where derived strategies is actively fostered

• Direct modeling		
• Counting	Comfort Zone	
• Deriving	Instructional Zone	Comfort Zone
• Number recall		Instructional Zone

Below is a four-phase instructional plan for developing a level of fluid recall of single-digit combinations.

- *Phase 1:* Accuracy (independent of time)
- *Phase 2:* Strategy Maturation (independent of time)
- *Phase 3:* Increasing Fluency Using Efficient Strategies via Fluency Interviews (time monitored)
- *Phase 4:* Fluency Recall (time dependent)

Phase One: Accuracy (independent of time)

The goal is to develop and assess a student's use of derived strategies. The timed element is removed. As I say to students, *I know you are good at counting on by ones* (addition and subtraction) *but I would rather you go slow and use these new strategies than count everything by ones. It will allow you to go faster later when the numbers get bigger.* While there is an element of "just eat your spinach. It's good for you," in that pitch, this consistent drumbeat provides students the emotional space and time to move to more sophisticated strategies and mathematical thinking that will aid them as they move forward.

This is pure instructional support. The main point is to be accurate first and foremost. No timing is used. General teacher observation is the formative assessment tool and that information is used to adjust instruction. Instructional activities that assist student in developing these strategies include:

- Using number strings, true/false questions, and open number sentences
- Get to Ten… One hundred… (and related landmarks) – Doubling, Tripling… a number; Splitting & sharing (Multiplication & Division)
- Card games (used for homework or in stations)

Phase Two: Strategy Maturation (independent of time)

More focused work around strategy development is completed in stations, independent practice, homework, or intervention settings to support students' use of the derived strategies. Students are monitored by the teacher to determine who is becoming more confident with derived strategies and who remains dependent on counting on/back strategies. When using more emergent strategies, teacher interruption becomes an instructional technique. *I know you can count on by ones, but how can you break the numbers apart and work with chunks of numbers rather than just one by one?* The goal is to scaffold the student to practice the derived strategies so that those strategies emerge as the student's new comfort level. The focus of this assessment is *what is a genuine developmental need of the student and what is habit?* If the determination is that the student's preference is one of habit, interrupt away! If the determination is that the student is developmentally young and deriving is still a genuine struggle, then pull back and let the student count on/back.

Work with multi-digit numbers likely has already occurred as in the case of second grade (addition and subtraction) and in third grade (multiplication and division). Drawing direct attention to where derived strategies are helpful in the students' work as well as highlighting when a student spontaneously transfers using a derived strategy in a multi-digit context is critical. *Oh! So, the strategy you use with smaller numbers is useful when working with larger numbers?* That attention to strategy is productive in helping to reinforce why and when to use the strategies.

Publicly capturing students' decompositions in written form also aids in the logic of how and where these strategies can be productively used. Example:

56 3
+ 37< 4
86
+ 4
90
+ 3
93

Question to pose: *So why did you break the seven into 3 and 4 and not something like 5 and 2 or 6 and 1?* Student response: *Because 4 and 6 make a ten so it gets me to 90 and 90 + 3 is 93!* Teacher response: *So, you used a "get to the next ten strategy"?* Student response: *Yes.*

Showing how and where a number was decomposed allows others in the class listening to the student share the thinking used to solve the problem. While an informal means of representation, is critical to help in the comprehension of where the 4 and 3 came from and the purpose of using *that single-digit combination* compared to other options. At this stage, the focus is on maturation of usage so that deriving becomes the comfort level for students independent of any time pressure.

Phase Three: Increasing Fluency Using Efficient Strategies via Fluency Interviews (time monitored)

General observation notes from listening to student responses to the warm-ups and related instructional tasks being used to develop derived strategies provides the teacher clear indication of who is gaining fluency and who is still developing consistent usage. It is at this point where conducting fluency checks with individual students is essential. This is an individual fluency assessment for second and third graders. This is a similar fluency check but for use with multiplication combinations for third and fourth graders. Use it with only those students of whom you have questions about how consistent they are with the range of combinations. If you know someone is consistently fluent, there is no need to assess every student individually. Use your time strategically.

Students may be fluid with many combinations but need to calculate some. The assessment protocol seeks to determine, if they are calculating, what strategies are they falling back on? Are they still relying on counting on/back by ones (or skip counting in the case of multiplication) or are they using derived strategies but are just more methodical working them through. Your follow-up instructional decisions are very different if the student is relying on the former or the latter. Follow-up with students in targeted instructional settings at this stage rather than with the whole class.

Phase Four: Fluency Recall (Time dependent)

It is only at this stage, with either those students who are demonstrating fluency in their general work or with the whole class that a timed test is used. But it is not used in the classic high-stress context that most of us grew up with. In order to elevate the use of derived strategies to the front of students' brains, I project the test up on the screen and ask (See figure 4.),

- *Which are the easy ones? Why are they so easy?*
- *Which are some that you have to slow down and think about? What strategies besides counting one by one (or skip counting) that we could use to save ourselves some time?*
- *Which of the easier ones could we use to figure out the harder ones?*

Figure 4: Example of Items found on a timed test projected before the assessment is initiated

0	5	2	6	9	5	3	3	8
x 9	x 8	x 7	x 5	x 6	x 2	x 1	x 0	x10
9	1	8	9	3	7	8	2	5
x 3	x 9	x 8	x 7	x 5	x 6	x 2	x 1	x 0
3	8	2	0	3	7	3	7	5
x 4	x 3	x 9	x 8	x 7	x 5	x 6	x 2	x 1

The purpose is to activate prior knowledge. I don't want students to revert under the stress of the assessment to using low level strategies. Besides elevating the strategies to the students' consciousness, various algebraic properties are reviewed. A timer is set once students begin work. When a student is done, I record the time needed on the page. That particular student then quietly reads or draws while others continue. If someone is still working after 10 minutes, all are stopped as that gives me the assessment information that I need. From an assessment standpoint, the following criteria is used.

- What is the accuracy relative to the time taken?
    - Accurate and fluid
    - Accurate and methodical (an indication of lots of deriving, just not at the recall level)
    - Accurate and labored (an indication of excessive counting)
    - Fast and inaccurate (lots of errors)
    - Labored and inaccurate

A message conveyed to students both before and after such assessments as well as during instruction is, *I would rather you be slow and accurate, and use deriving strategies, than fast and sloppy.* Speed for the sake of itself is not emphasized. Strategy development and its extension to the multi-digit range is. If it takes a student 6 minutes to be 100% accurate with a large set of combinations rather than 5 minutes, and derived strategies are being comfortably drawn upon, I am not concerned as a teacher. Such a student should not be held out of enriched mathematical instruction based on such a measure. *What makes a strong mathematician is the ability to persist, be flexible in one's thinking, and agile in the strategies one draws upon to work through the problem-solving process.* That is the message to be conveyed. Those are the habits of mind to be nurtured over time.

Summary

An important research finding to remember is students do not need to be fluent with single-digit combinations before moving on to solving multi-digit number combinations (Carpenter, et al., 2015). Working with multi-digit number combinations, in fact, leads to more fluency recall. Proficiency and procedural competency arise in combination with a student's conceptual understanding and number sense developed through working with multi-digit combinations. Taking time *over the arc of a school year* to nurture derived strategies so that those strategies become students' comfort zone leading not only to fluency but to important algebraic skills. Many math standards are addressed through this instructional approach.

Reflection Point 4

Take a moment with your colleagues to summarize what has been said in Part 4 and the linked articles about different instructional tasks along with the two assessment links.

- ☐ How does this emphasis on derived strategies mirror, or not, how you personally developed fluency with certain single-digit combinations?
- ☐ How does this approach towards developing fluency with derived strategies resonate with how you have seen

your students thinking in previous years?

- ☐ Would need to be adapted, supplemented, and or replaced in the school's current practice with the implementation policy put forward here?

Part V - Meeting the Standards

Developing More Than One Standard at a Time

A Common Core Standard for second grade states:

> *Fluently add and subtract within 20 using mental strategies. By the end of Grade 2, know from memory all sums of two one-digit numbers.*

But also at second grade are the following standards:

> *Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.*
>
> *Explain why addition and subtraction strategies work, using place value and the properties of operations.*

Common Core also emphasizes these mathematical practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Consider the following number string and notice how the intent of the design and the possible public conversation that may be generated meet all of the standards above while also developing fluent recall of single-digit combinations.

7 + 3	*What is 7 + 3?*
7 + 5	*Use what you know about 7+3 to figure out 7+5*
17 + 5	*Use what you know about getting to the 'next ten' to solve 17 + 5*
27 + 15 = 27 + 10 + 3 + ☐	*What goes in the box to make the number sentence true? Can the strategies used in the earlier math expressions help us with this equation?*
10 - 2	*If you are at 10, go back 2. Where would you be?*
14 - 6 = 14 - (4 + ☐)	*What would go in the box to make the equation true and why?*
14 - 4 = 14 - 4 - 2	*Is this equation true or false?* Follow-up question once an answer is given… *How is this the same or different from the previous equation?*
34 - 16	*Use strategies just described by each other to solve this expression. How can you break the numbers apart to make them easier to work with?*

Reflection Point 5

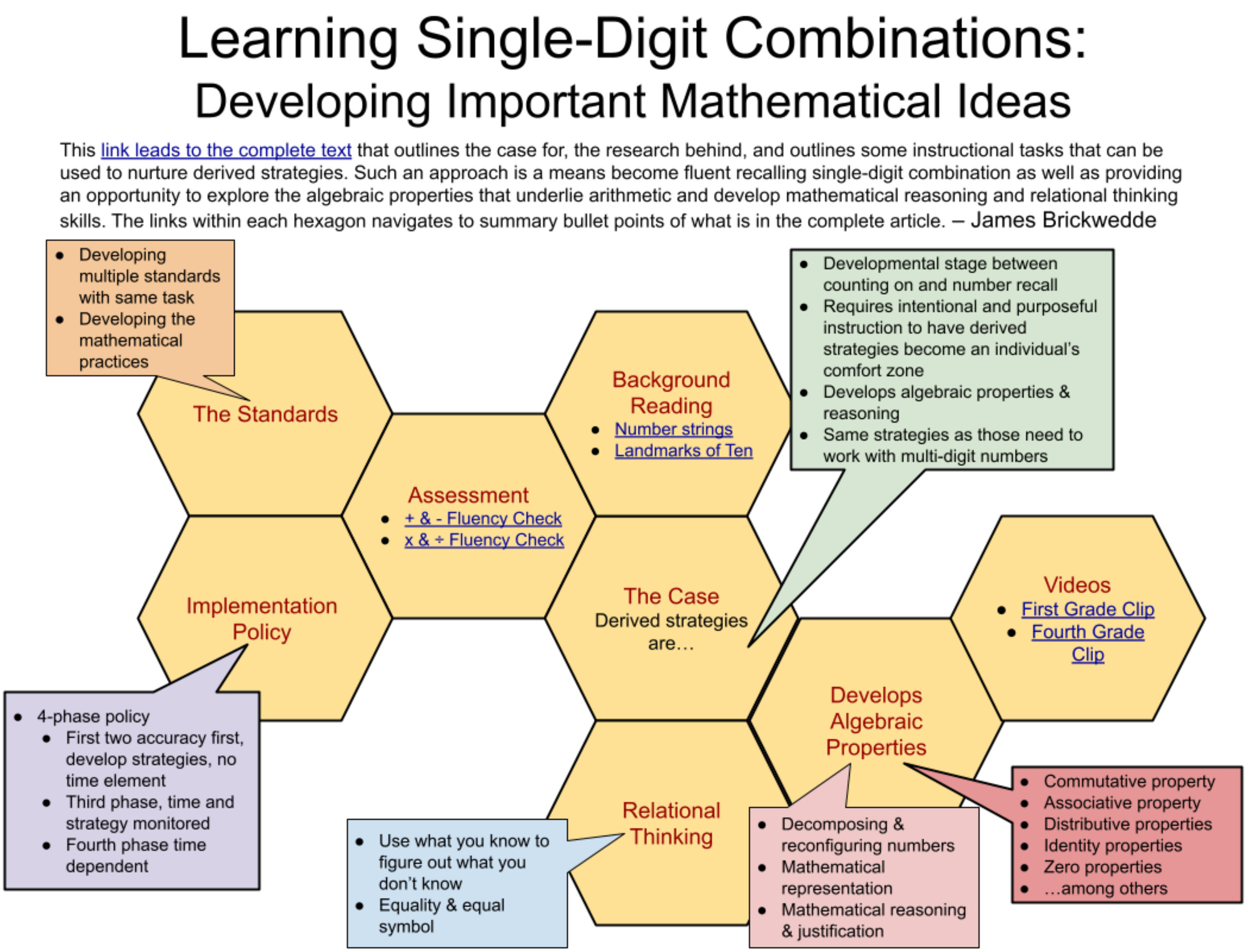
- ☐ Do the strings above address more than one standard?
- ☐ How are the mathematical practices addressed when enacting these two strings as part of a lesson?

Part VI - Additional Ways to Use Single-Digit Combinations to Foster Mathematical Thinking

Single-digit combinations can be both reviewed as well as used to stimulate rich mathematical conversations elsewhere in instruction. Consider some of the following and consider the various standards that might also be addressed if these are presented to your students.

- A difference between addition and subtraction
  - o 7 + 9 = 6 + 10 T/F? *reviews relational thinking, decomposing a number to then re-associate*
  - o 11 – 7 = 10 – 6 T/F? *It worked in addition, why not subtraction? Subtraction is not associative*
  - o 11 – 7 = 10 – ___ *What goes on the blank to make it true? The difference between two numbers Which of these two expressions would you rather work with? Oh, so if you don't like the number combination I give you, you can change it to make it easier to work with?*
- Factoring and the associative property of multiplication
  - o 4 x 6 = 3 x 8 *Why do these two expressions have the same product?*
- Identity property of multiplication
  - o 24 ÷ 6 = 48 ÷ 12 *Why do these two expressions have the same quotient?*

Graphic Organizer - To access an interactive version of this graphic organizer, click on this link.



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